

A generalization of product construction of multimagic squares

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Abstract

In this paper, constructions of multimagic squares are investigated. Diagonal Latin squares and Kronecker products are used to get some constructions of multimagic squares. Consequently, some new families of compound multimagic squares are obtained.

Keywords: Multimagic square, Latin square, Kronecker product.

1 Introduction

An $n \times n$ matrix A consisting of n^2 integers is a *general magic square of order n* if the sum of n elements in each row, each column and each of two diagonals are the same. The sum is the *magic number*. A general magic square of order n is a *magic square*, denoted by $MS(n)$, if the entries are n^2 consecutive integers. Usually, the (i, j) entry of a matrix A is denoted by $a_{i,j}$. A lot of work has been done on magic squares ([1–3, 6, 8, 13]).

Let t be a positive integer. A general magic square M is a *general t -multimagic square* if $M^{*1}, M^{*2}, \dots, M^{*t}$ are all general magic squares, where $M^{*e} = (m_{i,j}^e)$, $e = 1, 2, \dots, t$. A general t -multimagic square of order n is a *t -multimagic square*, denoted by $MS(n, t)$, if the entries are n^2 consecutive integers. Usually, a 2-multimagic square is called a *bimagic square* and a 3-multimagic square is called a *trimagic square*.

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In 2007, Derksen, Eggermont and van den Essen [9] used the matrices over rings to provide a constructive procedure. Recently, Chen and Li [7] introduced a magic pair of Latin squares to get a construction of bimagic squares. Zhang, Lei and Chen [16, 17] used large sets of orthogonal arrays to get a construction of multimagic squares.

In this paper, some new constructions of multimagic squares are provided by using diagonal Latin squares with Kronecker products.

A *Latin square* of order m , denoted by $LS(n)$, is an $m \times m$ array such that every row and every column is a permutation of an n -set S . A Latin square over S is called *diagonal* if it has the property that the main diagonal and back diagonal are both permutations of S . Two $LS(n)$ s are called *orthogonal* if each symbol in the first square meets each symbol in the second square exactly once when they are superposed.

Let $I_n = \{0, 1, \dots, n-1\}$ and let $J_{m \times n}$ be a matrix with the entries being all 1s. For an $MS(n, t)$ A , if the smallest entry integer is s , then it is readily verified that $A - sJ_{n \times n}$ is an $MS(n, t)$ over I_{n^2} . In the sequel, the element set of an $MS(n, t)$ is always taken to be I_{n^2} . For an $m \times n$ matrix A , the rows and columns of A are indexed by I_m and I_n , respectively.

Given an $m \times n$ matrix A and a $p \times q$ matrix B , the *Kronecker product* $A \otimes B$ is the $mp \times nq$ matrix given by $A \otimes B = (a_{i,j}B)$, where $a_{i,j}B = (a_{i,j}b_{s,t})$, $i \in I_m$, $j \in I_n$, $s \in I_p$, $t \in I_q$. For an $MS(m)$ A and an $MS(n)$ B , we define a *product* of A, B as follows.

$$A * B = n^2 A \otimes J_{n \times n} + J_{m \times m} \otimes B,$$

i.e.,

$$A * B = \begin{pmatrix} n^2 a_{0,0} J_{n \times n} + B & n^2 a_{0,1} J_{n \times n} + B & \cdots & n^2 a_{0,m-1} J_{n \times n} + B \\ n^2 a_{1,0} J_{n \times n} + B & n^2 a_{1,1} J_{n \times n} + B & \cdots & n^2 a_{1,m-1} J_{n \times n} + B \\ \vdots & \vdots & \vdots & \vdots \\ n^2 a_{m-1,0} J_{n \times n} + B & n^2 a_{m-1,1} J_{n \times n} + B & \cdots & n^2 a_{m-1,m-1} J_{n \times n} + B \end{pmatrix}.$$

It was proved in [11] that $A * B$ is an $MS(mn)$. Such a magic square is also called a *compound* magic square or a *composite* magic square in earlier papers such as [3, 9] and recent papers such as [12]. Further, Derksen et al [9] proved that if A is an $MS(m, t)$ and B is an $MS(n, t)$, then the product $A * B$ is an $MS(mn, t)$.

The main purpose of this paper is to generalize the product construction to get more composite multimagic squares. Some new constructions of multimagic squares are provided in Section 2, and some new families of bimagic squares and trimagic squares are given in Section 3 and Section 4, respectively.

2 Constructions of multimagic squares

Constructions of multimagic squares are investigated in this section. In the product construction of multimagic squares, one can find that $J_{m \times m} \otimes B$ is a general t -multimagic square of order mn whenever B is an $MS(n, t)$. We shall prove that the conclusion is still true if $J_{m \times m} \otimes B$ is replaced by a general t -multimagic square \mathcal{B} as follows:

$$\mathcal{B} = \begin{pmatrix} B_{0,0} & B_{0,1} & \cdots & B_{0,m-1} \\ B_{1,0} & B_{1,1} & \cdots & B_{1,m-1} \\ \vdots & \vdots & \vdots & \vdots \\ B_{m-1,0} & B_{m-1,1} & \cdots & B_{m-1,m-1} \end{pmatrix},$$

where $B_{u,v}$ is an $MS(n, t-1)$, $u, v \in I_m$. Such a partitioned general t -multimagic square \mathcal{B} is denoted by $PGMS(mn, t)$. We have the following.

Construction 2.1. *If there is an $MS(m, t)$ and there is a $PGMS(mn, t)$, then there is an $MS(mn, t)$.*

Proof. Let A be an $MS(m, t)$ and let \mathcal{B} be a $PGMS(mn, t)$, $\mathcal{B} = (B_{u,v})$, $B_{u,v} = (b_{r,s}^{(u,v)})$, $u, v \in I_m$, $r, s \in I_n$, and let

$$C = n^2 A \otimes J_{n \times n} + \mathcal{B},$$

i.e.,

$$c_{i,j} = n^2 a_{u,v} + b_{r,s}^{(u,v)}, i = nu + r, j = nv + s, u, v \in I_m, r, s \in I_n.$$

We shall prove that C is an $MS(mn, t)$.

Since $\{a_{u,v} | u, v \in I_m\} = I_{m^2}$ and $\{b_{r,s}^{(u,v)} | r, s \in I_n\} = I_{n^2}$, $u, v \in I_m$, we have

$$\{c_{i,j} | i, j \in I_{mn}\} = \{n^2 a_{u,v} + b_{r,s}^{(u,v)} | u, v \in I_m, r, s \in I_n\} = I_{(mn)^2},$$

which means that the $(mn)^2$ entries of C are $0, 1, \dots, (mn)^2 - 1$.

For an $MS(n, t)$ A , the magic number of A^{*e} is $S_e(n)$, where

$$S_e(n) = \frac{\sum_{k \in I_{n^2}} k^e}{n}, e = 1, 2, \dots, t.$$

It remains to prove that for each $e = 2, 3, \dots, t$,

$$\begin{aligned} \sum_{j=0}^{mn-1} c_{i,j}^e &= S_e(mn), i \in I_{mn}; \quad \sum_{i=0}^{mn-1} c_{i,j}^e = S_e(mn), j \in I_{mn}; \\ \sum_{i=0}^{mn-1} c_{i,i}^e &= S_e(mn); \quad \sum_{i=0}^{mn-1} c_{i,mn-1-i}^e = S_e(mn). \end{aligned}$$

In fact, by the definition of C , for each $e = 2, 3, \dots, t$, we have

$$\sum_{j=0}^{mn-1} c_{i,j}^e = \sum_{k=0}^e \binom{e}{k} n^{2(e-k)} \sum_{v=0}^{m-1} a_{u,v}^{e-k} \sum_{s=0}^{n-1} (b_{r,s}^{(u,v)})^k.$$

By hypothesis, $\sum_{v=0}^{m-1} a_{u,v}^e = S_e(m)$, $u \in I_m$, $e = 2, 3, \dots, t$, and $\sum_{s=0}^{n-1} (b_{r,s}^{(u,v)})^e = S_e(n)$, $r \in I_n$, $e = 2, 3, \dots, t-1$.

If $e < t$, we have

$$\sum_{j=0}^{mn-1} c_{i,j}^e = \sum_{k=0}^e \binom{e}{k} n^{2(e-k)} S_{e-k}(m) S_k(n).$$

If $e = t$, since B is a general t -multimagic square, $\sum_{v=0}^{m-1} \sum_{s=0}^{n-1} (b_{r,s}^{(u,v)})^t = m S_t(n)$, $s \in I_n$, we have

$$\begin{aligned} \sum_{j=0}^{mn-1} c_{i,j}^e &= \sum_{k=0}^{t-1} \binom{t}{k} n^{2(t-k)} \sum_{v=0}^{m-1} a_{u,v}^{t-k} \sum_{s=0}^{n-1} (b_{r,s}^{(u,v)})^k + \sum_{v=0}^{m-1} \sum_{s=0}^{n-1} (b_{r,s}^{(u,v)})^t \\ &= \sum_{k=0}^{t-1} \binom{t}{k} n^{2(t-k)} S_{t-k}(m) S_k(n) + m S_t(n), \\ &= \sum_{k=0}^t \binom{t}{k} n^{2(t-k)} S_{t-k}(m) S_k(n). \end{aligned}$$

Therefore for each $e (2 \leq e \leq t)$, we have

$$\sum_{j=0}^{mn-1} c_{i,j}^e = \sum_{k=0}^e \binom{e}{k} n^{2(e-k)} S_{e-k}(m) S_k(n).$$

The right hand of the above equality only depends on m, n, e , denoted by $N(m, n, e)$.

Therefore

$$\sum_{i=0}^{mn-1} \sum_{j=0}^{mn-1} c_{i,j}^e = mn N(m, n, e).$$

On the other hand, we have

$$\sum_{i=0}^{mn-1} \sum_{j=0}^{mn-1} c_{i,j}^e = \sum_{d=0}^{(mn)^2-1} d^e = mn S_e(mn).$$

which implies $N(m, n, e) = S_e(mn)$, thus $\sum_{j=0}^{mn-1} c_{i,j}^e = S_e(mn)$.

Similarly, one can prove that

$$\sum_{i=0}^{mn-1} c_{i,j}^e = S_e(mn), j \in I_{mn}, \sum_{i=0}^{mn-1} c_{i,i}^e = \sum_{i=0}^{mn-1} c_{i,mn-1-i}^e = S_e(mn).$$

Therefore, C is an $MS(mn, t)$. □

In the following, we shall showed that a $\text{PGMS}(mn, t)$ can be obtained by using a set of complementary t -multimagic squares and a diagonal Latin square.

Let B_0, B_1, \dots, B_{m-1} be m $\text{MS}(n, t)$ s, where $B_s = (b_{i,j}^{(s)})$, $i, j \in I_n$, $s \in I_m$. They are *complementary* if the following four conditions are satisfied.

$$(R1) \quad \sum_{s \in I_m} \sum_{j \in I_n} (b_{i,j}^{(s)})^{t+1} = mS_{t+1}(n), i \in I_n;$$

$$(R2) \quad \sum_{s \in I_m} \sum_{i \in I_n} (b_{i,j}^{(s)})^{t+1} = mS_{t+1}(n), j \in I_n;$$

$$(R3) \quad \sum_{s \in I_m} \sum_{i \in I_n} (b_{i,i}^{(s)})^{t+1} = mS_{t+1}(n);$$

$$(R4) \quad \sum_{s \in I_m} \sum_{i \in I_n} (b_{i, n-1-i}^{(s)})^{t+1} = mS_{t+1}(n).$$

The set of B_0, B_1, \dots, B_{m-1} is denoted by $m\text{-SCMS}(n, t)$. Usually, t omitted if $t = 1$.

As an example, a set of 2 complementary magic squares is listed below.

Example 1 Let

$$B_0 = \begin{pmatrix} 2 & 12 & 5 & 11 \\ 9 & 7 & 14 & 0 \\ 15 & 1 & 8 & 6 \\ 4 & 10 & 3 & 13 \end{pmatrix}, B_1 = \begin{pmatrix} 1 & 15 & 6 & 8 \\ 10 & 4 & 13 & 3 \\ 12 & 2 & 11 & 5 \\ 7 & 9 & 0 & 14 \end{pmatrix}.$$

Then $\{B_0, B_1\}$ is a 2-SCMS(4). In fact, it is easily checked that B_0, B_1 are $\text{MS}(4)$ s and

$$\begin{aligned} \sum_{j \in I_4} (b_{i,j}^{(0)})^2 + \sum_{j \in I_4} (b_{i,j}^{(1)})^2 &= 2S_2(4) = 620, i \in I_4, \\ \sum_{i \in I_4} (b_{i,j}^{(0)})^2 + \sum_{i \in I_4} (b_{i,j}^{(1)})^2 &= 620, j \in I_4, \\ \sum_{i \in I_4} (b_{i,i}^{(0)})^2 + \sum_{i \in I_4} (b_{i,i}^{(1)})^2 &= 620, \\ \sum_{i \in I_4} (b_{i,3-i}^{(0)})^2 + \sum_{i \in I_4} (b_{i,3-i}^{(1)})^2 &= 620. \end{aligned} \quad \square$$

Construction 2.2. Let $t \geq 2$. If there is an $\text{MS}(m, t)$ and there is an $m\text{-SCMS}(n, t-1)$, then there is an $\text{MS}(mn, t)$.

Proof. By hypothesis we know that $m \geq 4$ since there is no $\text{MS}(3, 2)$ (see [4]). By Reference [10] we know that a diagonal $\text{LS}(m)$ exists if and only if $m \geq 4$. Let D be a diagonal $\text{LS}(m)$ over I_m . Let $\{B_0, B_1, \dots, B_{m-1}\}$ be an $m\text{-SCMS}(n, t-1)$, where $B_s = (b_{x,y}^{(s)})$, $x, y \in I_n$, $s \in I_m$. and let $P_m(u, v)$ be a matrix of order m with the (u, v) entry 1 and all other entries 0s. Define a matrix \mathcal{B} of order mn as follows.

$$\mathcal{B} = \sum_{u,v \in I_m} P_m(u, v) \otimes B_{d_{u,v}}.$$

We shall show that \mathcal{B} is an $\text{PGMS}(mn, t)$.

In fact, if we denote $\mathcal{B} = (b_{i,j})_{mn \times mn}$, $i, j \in I_{mn}$, then

$$b_{i,j} = b_{x,y}^{(d_{u,v})}, i = nu + x, j = nv + y, u, v \in I_m, x, y \in I_n.$$

Since D is a diagonal LS(m), we have $\{d_{u,v} | v \in I_m\} = I_m, u \in I_m$. For each $i \in I_{mn}$ and for each $e \in \{1, 2, \dots, t\}$,

$$\sum_{j=0}^{mn-1} (b_{i,j})^e = \sum_{v=0}^{m-1} \sum_{y=0}^{n-1} (b_{x,y}^{(d_{u,v})})^e = \sum_{v=0}^{m-1} \sum_{y=0}^{n-1} (b_{x,y}^{(v)})^e.$$

If $e < t$, then $\sum_{v=0}^{m-1} \sum_{y=0}^{n-1} (b_{x,y}^{(v)})^e = m \sum_{y=0}^{n-1} (b_{x,y}^{(v)})^e = mS_e(n)$ since B_0, B_1, \dots, B_{m-1} are MS($n, t-1$)s. If $e = t$, then $\sum_{v=0}^{m-1} \sum_{y=0}^{n-1} (b_{x,y}^{(v)})^t = mS_t(n)$ since B_0, B_1, \dots, B_{m-1} are complementary.

Thus $\sum_{j=0}^{mn-1} (b_{i,j})^e = mS_e(n)$ for each $e \in \{1, 2, \dots, t\}$.

Similarly, for each $e \in \{1, 2, \dots, t\}$ we have

$$\begin{aligned} \sum_{i=0}^{mn-1} (b_{i,j})^e &= mS_t(n), j \in I_{mn}; \\ \sum_{i=0}^{mn-1} (b_{i,i})^e &= mS_e(n), \\ \sum_{i=0}^{mn-1} (b_{i,mn-1-i})^e &= mS_e(n). \end{aligned}$$

So, \mathcal{B} is a PGMS(mn, t). By Construction 2.1 the proof is completed. \square

We use the following example to illustrate the proofs of Construction 2.2.

Example 2 Let $B_i = B_0, B_{i+1} = B_1, i = 2, 4, 6$, where B_0, B_1 are given in Example 1. It is readily checked that $\{B_0, B_1, \dots, B_7\}$ is an 8-SCMS(4). A DLS(8) over I_8 is listed below.

$$D = \begin{pmatrix} 0 & 3 & 6 & 5 & 4 & 7 & 2 & 1 \\ 1 & 2 & 7 & 4 & 5 & 6 & 3 & 0 \\ 5 & 6 & 3 & 0 & 1 & 2 & 7 & 4 \\ 4 & 7 & 2 & 1 & 0 & 3 & 6 & 5 \\ 2 & 1 & 4 & 7 & 6 & 5 & 0 & 3 \\ 3 & 0 & 5 & 6 & 7 & 4 & 1 & 2 \\ 7 & 4 & 1 & 2 & 3 & 0 & 5 & 6 \\ 6 & 5 & 0 & 3 & 2 & 1 & 4 & 7 \end{pmatrix}.$$

Let $\mathcal{B} = \sum_{u,v \in I_8} P_8(u, v) \otimes B_{d_{u,v}}$, i.e.,

3 New families of bimagic squares

In this section, some new families of bimagic squares are provided by constructing a family of complementary magic squares.

A *Kotzig array* of size $m \times n$, denoted by $KA(m, n)$, is an $m \times n$ array such that each row is a permutation of $\{0, 1, 2, \dots, n-1\}$, each column has the same sum. Clearly, the common sum must be $\frac{1}{2}m(n-1)$, which is an integer only if $m(n-1)$ is even.

Using Kotzig array and diagonal orthogonal Latin squares we have the following.

Lemma 3.1. *If there is a $KA(m, n)$ and there is a pair of diagonal orthogonal $LS(n)$ s, then there is an m -SCMS(n).*

Proof. Let K be an $KA(m, n)$, and let A, B be a pair of orthogonal diagonal $LS(n)$ s over I_n . Let

$$B_s = (b_{i,j}^{(s)}), \quad b_{i,j}^{(s)} = k_{s,b_{i,j}}, \quad i, j \in I_n, \quad s \in I_m.$$

Then A, B_s are orthogonal diagonal $LS(n)$ s for each $s \in I_m$. Let

$$C_s = (c_{i,j}^{(s)}), \quad c_{i,j}^{(s)} = na_{i,j} + b_{i,j}^{(s)}, \quad i, j \in I_n, \quad s \in I_m.$$

It is easy to see that C_s is an $MS(n)$ for each $s \in I_m$. We shall prove that $\{C_0, C_1, \dots, C_{m-1}\}$ is an m -SCMS(n).

In fact, for any $i \in I_n$, we have

$$\begin{aligned} \sum_{s=0}^{m-1} \sum_{j=0}^{n-1} (c_{i,j}^{(s)})^2 &= \sum_{s=0}^{m-1} \sum_{j=0}^{n-1} (na_{i,j} + b_{i,j}^{(s)})^2 \\ &= n^2 \sum_{s=0}^{m-1} \sum_{j=0}^{n-1} a_{i,j}^2 + \sum_{s=0}^{m-1} \sum_{j=0}^{n-1} (b_{i,j}^{(s)})^2 + 2n \sum_{j=0}^{n-1} a_{i,j} \sum_{s=0}^{m-1} b_{i,j}^{(s)}. \end{aligned}$$

By the definition of a $KA(m, n)$, we have

$$\sum_{s=0}^{m-1} b_{i,j}^{(s)} = \sum_{s=0}^{m-1} k_{s,b_{i,j}} = m(n-1)/2, \quad j \in I_n.$$

Since A, B_s are Latin squares, we have $\sum_{j=0}^{n-1} a_{i,j} = n(n-1)/2$ and

$$\sum_{j=0}^{n-1} (a_{i,j})^2 = \sum_{j=0}^{n-1} (b_{i,j}^{(s)})^2 = n(n-1)(2n-1)/6, \quad s \in I_m.$$

Noting that $S_2(n) = n(n^2-1)(2n^2-1)/6$, from above we have

$$\begin{aligned} \sum_{s=0}^{m-1} \sum_{j=0}^{n-1} (c_{i,j}^{(s)})^2 &= m(n^2+1)n(n-1)(2n-1)/6 + n^2(n-1)m(n-1)/2 \\ &= mn(n^2-1)(2n^2-1)/6 = mS_2(n). \end{aligned}$$

Similarly, we have

$$\sum_{s=0}^{m-1} \sum_{i=0}^{n-1} (c_{i,j}^{(s)})^2 = mS_2(n), \quad j \in I_n;$$

$$\sum_{s=0}^{m-1} \sum_{i=0}^{n-1} (c_{i,i}^{(s)})^2 = \sum_{s=0}^{m-1} \sum_{i=0}^{n-1} (c_{i,n-1-i}^{(s)})^2 = mS_2(n).$$

So, $\{C_0, C_1, \dots, C_{m-1}\}$ is an m -SCMS(n). □

Example 4 Let

$$A = \begin{pmatrix} 1 & 3 & 0 & 2 & 4 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 0 & 2 & 4 & 1 \\ 4 & 1 & 3 & 0 & 2 \\ 0 & 2 & 4 & 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 & 4 & 0 \\ 3 & 4 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \\ 4 & 0 & 1 & 2 & 3 \end{pmatrix}.$$

Then A, B is a pair of orthogonal diagonal LS(5)s. A KA(3, 5) K is listed below.

$$K = \begin{pmatrix} 4 & 1 & 3 & 0 & 2 \\ 0 & 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 & 0 \end{pmatrix}.$$

Let $B_s = (b_{i,j}^{(s)})$, $b_{i,j}^{(s)} = k_{s,b_{i,j}}$, $i, j \in I_5$, $s \in I_3$, i.e.,

$$B_0 = \begin{pmatrix} 1 & 3 & 0 & 2 & 4 \\ 0 & 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 0 & 2 \\ 3 & 0 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 0 \\ 3 & 4 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \\ 4 & 0 & 1 & 2 & 3 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 4 & 1 & 3 & 0 & 2 \\ 3 & 0 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 0 \\ 1 & 3 & 0 & 2 & 4 \\ 0 & 2 & 4 & 1 & 3 \end{pmatrix}.$$

Let $C_s = (c_{i,j}^{(s)})$, $c_{i,j}^{(s)} = na_{i,j} + b_{i,j}^{(s)}$, $i, j \in I_5$, $s \in I_3$, i.e.,

$$C_0 = \begin{pmatrix} 6 & 18 & 0 & 12 & 24 \\ 10 & 22 & 9 & 16 & 3 \\ 19 & 1 & 13 & 20 & 7 \\ 23 & 5 & 17 & 4 & 11 \\ 2 & 14 & 21 & 8 & 15 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 6 & 17 & 3 & 14 & 20 \\ 13 & 24 & 5 & 16 & 2 \\ 15 & 1 & 12 & 23 & 9 \\ 22 & 8 & 19 & 0 & 11 \\ 4 & 10 & 21 & 7 & 18 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 9 & 16 & 3 & 10 & 22 \\ 13 & 20 & 7 & 19 & 1 \\ 17 & 4 & 11 & 23 & 5 \\ 21 & 8 & 15 & 2 & 14 \\ 0 & 12 & 24 & 6 & 18 \end{pmatrix}.$$

It is readily checked that $\{C_0, C_1, C_2\}$ is a 3-SCMS(5). □

For the existence of Kotzig array, Wallis [15] proved the following.

Lemma 3.2. ([15]) *There is a KA(m, n) if and only if $m > 1$ and $m(n-1)$ is even.*

For the existence of orthogonal diagonal Latin squares, Brown et al [5] proved the following.

Lemma 3.3. ([5]) *A pair of orthogonal diagonal LS(n)s exists if and only if $n \notin \{2, 3, 6\}$.*

By Lemma 3.1, Lemma 3.2 and Lemma 3.3 we have

Lemma 3.4. *There exists an m -SCMS(n) whenever $m(n-1)$ is even and $m > 1$, and $n \notin \{2, 3, 6\}$.*

By Construction 2.2 and Lemma 3.4, we have the following.

Lemma 3.5. *If there exists an $MS(m, 2)$, then there exists an $MS(mn, 2)$ whenever $m(n-1)$ is even and $n \notin \{2, 3, 6\}$.*

Let $\Omega_1 = \{n | 8 \leq n \leq 64\}$. An $MS(n, 2)$ for each $n \in \Omega_1$ is listed in Boyer's website [4]. Recently, Chen and Li [7] gave two families of bimagic squares. They showed that there exists an $MS(n, 2)$ for each $n \in \Omega_2 \cup \Omega_3$, where $\Omega_2 = \{n | n = n_1 n_2, n_1 \equiv n_2 \pmod{2}, n_1, n_2 \notin \{2, 3, 6\}\}$ and $\Omega_3 = \{n | n \equiv 0 \pmod{4}, n \geq 4\}$. These are summarized as follows.

Lemma 3.6. [4, 7] *There exists an $MS(n, 2)$ for $n \in \Omega_1 \cup \Omega_2 \cup \Omega_3$.*

Let $\Omega_4 = \{mn | mn > 64, 8 \leq m \leq 64, m \equiv 2 \pmod{4}, n \geq 5, n \equiv 1 \pmod{2}\}$, by using Lemma 3.5 and Lemma 3.6 we have

Theorem 3.7. *There exists an $MS(mn, 2)$ for $mn \in \Omega_4$.*

Proof. Let $mn \in \Omega_4$, there exists an $MS(m, 2)$ by Lemma 3.6. Since m is even, by Lemma 3.5, there exists an $MS(mn, 2)$. \square

Remark We should point out that $MS(mn, 2)$ s given in Theorem 3.7 are of orders $mn \equiv 2 \pmod{4}$, which are the new ones comparing to those given in Lemma 3.6.

4 New families of trimagic squares

In this section, complementary bimagic squares are considered and some new families of trimagic squares are obtained.

Lemma 4.1. *For even t , if there exists an $MS(n, t)$, then there exists a $2l$ -SCMS(n, t) for $l \geq 1$.*

Proof. Suppose that A is an $MS(n, t)$ with even t . Let $B = (n^2 - 1)J_{n \times n} - A$. It is readily checked that B is also an $MS(n, t)$. Note that $t + 1$ is odd, for each $i \in I_n$, we have

$$\begin{aligned} \sum_{j=0}^{n-1} (a_{i,j}^{t+1} + b_{i,j}^{t+1}) &= \sum_{j=0}^{n-1} (a_{i,j}^{t+1} + (n^2 - 1 - a_{i,j})^{t+1}) \\ &= \sum_{j=0}^{n-1} \sum_{k=0}^t \binom{t+1}{k} (-1)^k (n^2 - 1)^{t+1-k} a_{i,j}^k \\ &= \sum_{k=0}^t \binom{t+1}{k} (-1)^k (n^2 - 1)^{t+1-k} \sum_{j=0}^{n-1} a_{i,j}^k \\ &= \sum_{k=0}^t \binom{t+1}{k} (-1)^k (n^2 - 1)^{t+1-k} S_k(n), \end{aligned}$$

which is independent of i . Hence,

$$\sum_{j=0}^{n-1} (a_{i,j}^{t+1} + b_{i,j}^{t+1}) = \frac{1}{n} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (a_{i,j}^{t+1} + b_{i,j}^{t+1}) = \frac{1}{n} \left(\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} a_{i,j}^{t+1} + \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} b_{i,j}^{t+1} \right) = 2S_{t+1}(n).$$

Similarly, we have $\sum_{i=0}^{n-1} (a_{i,j}^{t+1} + b_{i,j}^{t+1}) = 2S_{t+1}(n), j \in I_n$, and

$$\sum_{i=0}^{n-1} (a_{i,i}^{t+1} + b_{i,i}^{t+1}) = 2S_{t+1}(n), \quad \sum_{i=0}^{n-1} (a_{i,n-1-i}^{t+1} + b_{i,n-1-i}^{t+1}) = 2S_{t+1}(n).$$

Thus, $\{A, B\}$ is a 2-SCMS(n, t). For $l \geq 1$, let $A_i = A, B_i = B, i \in I_l$. It is readily checked that $\{A_0, B_0, A_1, B_1, \dots, A_{l-1}, B_{l-1}\}$ is a $2l$ -SCMS(n, t). \square

By Lemma 4.1 and Lemma 3.6 we get the following.

Lemma 4.2. *Let l be a positive integer, then there exists a $2l$ -SCMS($n, 2$) for $n \in \Omega_1 \cup \Omega_2 \cup \Omega_3$.*

Some trimagic squares are listed in the Boyer's website [4].

Lemma 4.3. ([4]) *There exists an MS($n, 3$) for all $m \in \{12, 16, 24, 32, 40, 64, 128\}$.*

Lemma 4.4. *There exists an MS($2^m, 3$) for all $m \geq 4$.*

Proof. There exists an MS($2^m, 3$) for $m \in \{4, 5, 6, 7\}$ by Lemma 4.3. For any $m \geq 8$, we can write $m = 4n_1 + 5n_2 + 6n_3 + 7n_4$, where n_1, n_2, n_3, n_4 are nonnegative integers. By the product construction of multimagic squares mentioned in Section 1 we get an MS($2^m, 3$). \square

Lemma 4.5. *For even m and odd t , if there exists an MS(m, t) and there exists an MS($n, t-1$), then there exists an MS(mn, t).*

Proof. Since there exists an MS($n, t-1$), m is even and t is odd, there exists an m -SCMS($n, t-1$) by Lemma 4.1. Hence, there exists an MS(mn, t) by Construction 2.2. \square

Theorem 4.6. *There exists an MS($mn, 3$) for any $m \in M = \{12, 24, 40\} \cup \{2^l | l \geq 4\}$ and $n \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$.*

Proof. For any $m \in M = \{12, 24, 40\} \cup \{2^l | l \geq 4\}$, there exists an MS($m, 3$) by Lemma 4.3 and Lemma 4.4. For any $n \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$, there exists an MS($n, 2$) by Lemma 3.6 and Theorem 3.7. So there exists an MS($mn, 3$) by Lemma 4.5. \square

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